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DIAGRAMMATIC PROOF OF THE CONDITION OF FUNCTIONALITY IN COMPLEX FUNCTIONS.

By A. LATHAM BAKER, C. E., Ph. D., University of Rochester, Rochester, N. Y.

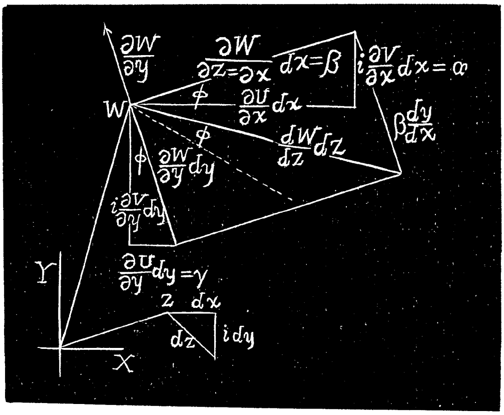
1°. If the given complex function $W(z)=W(x+iy)=U+iV$ has a derivative we may write $\frac{dW}{dz}=w$, which phrased for the Argand Diagram is: *The triangle determined by dW, dz is similar to that determined by w, 1.*

At the point z, suppose z to take the increment $\partial z=dx$, causing in W, the change

$$\frac{\partial W}{\partial z=dx}dx=\frac{\partial U}{\partial x}dx+i\frac{dV}{\partial x}dx$$

lying at an angle, say ϕ, with $\partial z=dx$. This is shown in the diagram.

Suppose z had taken the increment dz, causing in W the change $\frac{dW}{dz}dz$, making with dz the same angle ϕ.


$\frac{\partial W}{\partial y}$
$\frac{\partial W}{\partial z=\partial x}dx=\beta$
$i\frac{\partial V}{\partial x}dx=\alpha$
W
ϕ
$\frac{\partial U}{\partial x}dx$
$\beta\frac{dy}{dx}$
ϕ
$\frac{dW}{dz}dz$
ϕ
$\frac{\partial W}{\partial y}dy$
$i\frac{\partial V}{\partial y}dy$
$\frac{\partial U}{\partial y}dy=\gamma$
Y
z
dx
dz
idy
X

Since $\frac{\partial W}{\partial x}=\frac{dW}{dz}=\frac{w}{1}$, we have $\frac{dW}{\partial W}=\frac{dz}{\partial x}$ and $\frac{dW}{dz}dz$ will make with $\frac{\partial W}{\partial x}dx$ an angle whose secant is $\frac{dz}{dx}=\tan^{-1}\frac{dy}{dx}$.

Similarly when z takes the increment idy, $\frac{\partial W}{\partial z=dy}dy$ makes with the vertical the same angle ϕ. All these increments are shown in the diagram.

2°. Denoting the *lengths* of the lines in the diagram by Greek letters as shown, we get

$$\gamma=\beta\frac{dy}{dx}.\sin\beta=\beta\frac{dy}{dx}.\frac{\alpha}{\beta}=\alpha\frac{dy}{dx};$$

therefore

$$\frac{\gamma}{\alpha}=\left\{\frac{\frac{\partial U}{\partial y}dy}{\frac{\partial V}{\partial x}dx}\right\}=\frac{dy}{dx},$$

or introducing the Argand elements,

$$\frac{\frac{\partial U}{\partial y}dy}{i\frac{\partial V}{\partial x}dx}=\frac{idy}{dx}.$$

In the same manner

$$\left|\frac{\partial V}{\partial y}dy\right| = \beta\frac{dy}{dx}\cos\phi=\beta\frac{dy}{dx}\,\frac{\frac{\partial U}{\partial x}dx}{\beta}=\frac{dy}{dx}\frac{\partial U}{\partial x}dx,$$

or in the Argand Diagram

$$\frac{i\frac{\partial V}{\partial y}dy}{\frac{\partial U}{\partial x}dx}=\frac{idy}{dx}.$$

These two results entail as the results of functionality (Cauchy's conditions).

$$\frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}. \quad \frac{\partial V}{\partial y}dy=\frac{\partial U}{\partial x}dx.$$

3°. Or we might have said, since $\frac{\gamma}{dy}=\frac{\alpha}{dx}$ or $\left|\frac{\partial U}{\partial y}\right| = \left|i\frac{\partial V}{\partial x}\right|$.

Whence multiplying $i\frac{\partial V}{\partial x}$ by i revolves it into coincidence with $\frac{\partial U}{\partial y}$, that is, (see paragraph 4°) $-\frac{\partial V}{\partial x}=\frac{\partial U}{\partial y}$.

Similarly, $\left|i\frac{\partial V}{\partial y}\right| = \left|\frac{\partial U}{\partial x}\right|$.

Whence, $i\frac{\partial V}{\partial y}=i\frac{\partial U}{\partial x}$, or $\frac{\partial V}{\partial y}=\frac{\partial U}{\partial x}$.

4°. Since $\frac{\beta}{\delta}=\frac{dx}{dy}$, in the same way $\left|\frac{\partial W}{\partial x}\right| = \left|\frac{\partial W}{\partial y}\right|$. But $\frac{\partial W}{\partial x}$ was produced by dividing $\frac{\partial W}{\partial y}dy$ by dy. In the diagram dy is essentially negative, and remembering that division is doing to the operand what was done to the divisor to produce unity, we revolve $\frac{\partial W}{\partial y}dy$ through 180°, and change in the proper ratio, giving $\frac{\partial W}{\partial y}$ as shown in the diagram. The same remarks apply to $\frac{\partial W}{\partial x}$, but here the angle is zero.

Evidently revolving $\frac{\partial W}{\partial x}$ through 90° gives coincidence with $\frac{\partial W}{\partial y}$, or

$$i\frac{\partial W}{\partial x}=\frac{\partial W}{\partial y},$$

which is Riemann's condition of functionality.

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INTEGRATION OF ELLIPTIC INTEGRALS.

By G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

[Continued from October Number.]

Let $e_1=\frac{e^2}{2\pm e^2}$, $e_2=\frac{e_1^2}{2\pm e_1^2}$, $e_3=\frac{e_2^2}{2\pm e_2^2}$, ... , $e_n=\frac{e^2{}_{n-1}}{2\pm e^2{}_{n-1}}$.

$$[(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]^{\frac{1}{2}}=R,$$

$$[(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]^{\frac{1}{2}}=R_1,$$

$$[(1-e_2^2\sin^2 4\theta)(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]^{\frac{1}{2}}=R_2,$$

$$[(1-e_3^2\sin^2 8\theta)(1-e_2^2\sin^2 4\theta)(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]=R_3$$

. . . . . .

$$[(1-e_n^2\sin^2 2^n\theta)(1-e^2{}_{n-1}\sin^2 2^{n-1}\theta)\ldots.(1-e_3^2\sin^2 8\theta)$$

$$[(1-e_2^2\sin^2 4\theta)(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]\ \ldots R_n.$$

But $R=[1\pm e^2(\sin^2\theta+\cos^2\theta)+e^4\sin^2\theta\cos^2\theta]^{\frac{1}{2}}=[1\pm e^2+\frac{1}{4}e^4\sin^2 2\theta]^{\frac{1}{2}}$

$$=\tfrac{1}{2}[(2\pm e^2)^2-e^4\cos^2 2\theta]^{\frac{1}{2}}=\tfrac{1}{2}(2\pm e^2)[1-e_1^2\sin^2(\tfrac{1}{2}\pi-2\theta)]^{\frac{1}{2}}.$$

Let $\frac{1}{2}\pi-2\theta=\varphi$. Then $d\theta=-\frac{1}{2}d\varphi$.

$$\therefore\int_0^{\frac{1}{2}\pi}Rd\theta=\tfrac{1}{4}(2\pm e^2)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\surd(1-e_1^2\sin^2\varphi)d\varphi$$

$$=\tfrac{1}{2}(2\pm e^2)\int_0^{\frac{1}{2}\pi}\surd(1-e_1^2\sin^2\varphi)d\varphi=\tfrac{1}{2}(2\pm e^2)E(e_1,\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots\ldots\ldots(85).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{R}=\frac{2}{2\pm e^2}\int_0^{\frac{1}{2}\pi}\frac{d\varphi}{\surd(1-e_1^2\sin^2\varphi)}=\frac{2}{2\pm e^2}F(e_1,\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots(86).$$